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COMPUTATION OF THE CIRCULAR ERROR PROBABLE (CEP) AND CONFIDENCE INTERVALS IN BOMBING TESTS

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13. ABSTRACT (Maximum 200 words)

The objective of this report is to assess the performance of a radar bomb director (RBD) by computing its CEP (Circular Error Probable) on the basis of measurements of miss distances in actual bomb drops falling under an uncorrelated bivariate normal distribution. In addition, confidence intervals (CIs) for the CEP are established.

On the basis of these results and the assignment of a nominal CEP, one can also decide whether or not the statistical accuracy requirements of the RBD have been met, or whether further testing is required.

An available Fortran 95 computer program is described, which for a given set of bomb drops, generates an estimated CEP and three different CIs: a conventional interval, a minimum one, and when possible, a symmetric one. The acceptance or nonacceptance of the RBD's performance, or whether more testing is needed, is also an optional part of the output.

14. SUBJECT TERMS
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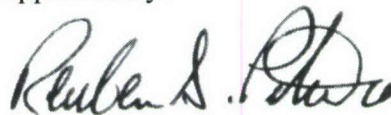| 17. SECURITY CLASSIFICATION OF REPORTS	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
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FOREWORD

The statistical analysis described in this report represents an extension of Dr. Milton Jarnagin's work described in a 1971 memorandum. The analysis is the basis for Fortran 95 software of important statistical functions that are not contained in the NSWC Library of Mathematics Subroutines. The software can be used in targeting studies, such as assessing the performance of a radar bomb director, where statistical confidence regions are required.

Dr. John Crigler (Q21) supplied the external distribution list, and Dr. Robert McDevitt (W32) helped clarify the presentation.

This document was reviewed by Robert G. Hill (W60), Head, Engineering and Command Environment Division.

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I. INTRODUCTION

One problem, of two under consideration, is to assess the performance of a radar bomb director (RBD) by estimating its Circular Error Probable (CEP) given in units of length,[1] based on measurements of miss distances from the target at $(0,0)$ in actual bomb drops that fall under an uncorrelated bivariate normal distribution (UBND) with zero means and unknown variances. Confidence intervals (CIs) for the true CEP (CEPT), will be needed, where the CEPT, by definition, is the radius, RT, of the circle centered at $(0,0)$, which contains 50%, PCEP $= 0.50$, of the distribution. The computer program, CONFREG, discussed in Section VI, allows arbitrary PCEP values in $(0,1)$.

In the second problem, it is assumed a radar contractor is to be awarded a bonus if the computed CEP is below an agreed nominal value (RTn, for example by 10%), and is to be penalized if over the RTn by a like percentage. Thus, it is important to know the level of confidence, under these circumstances, that can be placed on the CEP, R, computed from experimental data as an estimate of the RT.

In May 1971, Dr. M. P. Jarnagin of the Naval Weapons Laboratory at Dahlgren, Virginia, issued an informal memorandum [1] in which he described statistical procedures for addressing these problems. Because of limited computing facilities, his analysis used approximations throughout and, consequently, the work was limited in scope. Our objective is to replace the approximations with precise calculations and to generalize the results to some extent. In many parts of this report, excerpts are taken directly from his memorandum.

Other studies addressing the first problem have been carried out in [2] and [3]. Again, their analyses are based on approximations without error bounds rather than on precise calculations.

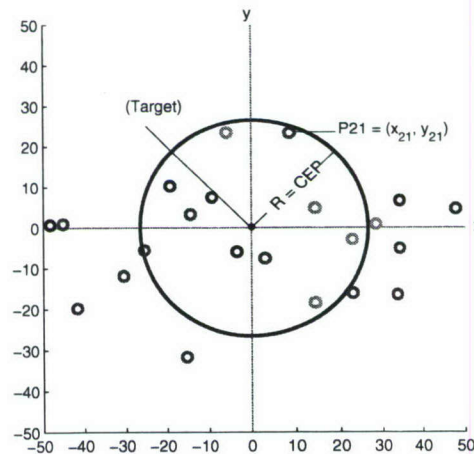A set of hypothetical individual bomb drops is shown in Figure 1.

Figure 1. Typical Bombing Pattern, n = 22

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[1] In this report, the unit of length will be meters.

Let n denote the total number of bomb drops. The target is a point target at $(0,0)$, the origin of coordinates. The flight path of the aircraft dropping each bomb is along the x-axis. The y-axis passes through the target, is perpendicular to the x-axis, and is positive above the x-axis. The x and y coordinates of the impact point of a given bomb drop are the components of the miss distance in range and deflection, respectively, in the usual terminology.

In Figure 1, $n = 22$, Bomb No. 21 hits at point P21 with coordinates (in meters) $x = 8.668$, $y = 23.298$.[2] The estimated CEP, or more briefly R, is the radius of the circle that contains 50% of the impact points. An indicated CEP in Figure 1 is $R = 26.5$. For such finite samples, R is not uniquely defined, but for a theoretical bivariate normal distribution, RT is a precisely defined unique number.

The assumption is made that the x-coordinates of the bomb drops, $\{x_i\} = x_1, x_2, \cdots, x_n$, constitute a random sample from a normal distribution with mean zero and with unknown standard deviation σ_x. Similarly, it is assumed that the y-coordinates of the bomb drops, $\{y_i\} = y_1, y_2, \cdots, y_n$, form a random sample from a normal distribution, independent of the distribution of the x_i's, with mean zero and unknown standard deviation σ_y. Thus, the impact points (x_i, y_i) are assumed to be from a UBND. The results in an actual test may not indicate zero means. That would imply a bias in the performance of the RBD that would penalize the contractor. Subsequently, we allow for nonzero means, which extends the model in [1].

The CEPT($= $ RT) is a function of the standard deviations σ_x and σ_y. We compute an estimate R for RT on the basis of the experimentally determined estimates s_x and s_y, for σ_x and σ_y, where

$$s_x \equiv \sqrt{\sum_{i=1}^{n} x_i^2 / f}, \quad s_y \equiv \sqrt{\sum_{i=1}^{n} y_i^2 / f}, \quad f = n, \quad (\text{Assume } \bar{x} = 0, \ \bar{y} = 0), \tag{1}$$

$$s_x \equiv \sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 / f}, \quad s_y \equiv \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2 / f}, \quad f = n - 1, \ \bar{x} \neq 0, \ \bar{y} \neq 0, \tag{2}$$

and where f denotes the number of degrees of freedom in the data. The means in (2) are given by

$$\bar{x} \equiv \sum_{i=1}^{n} x_i / n, \quad \bar{y} \equiv \sum_{i=1}^{n} y_i / n. \tag{3}$$

The pertinent function connecting the CEP and the estimated standard deviations taken from (1) is the Generalized Circular Error Function (GCE), $V(K, c)$, where $K = R/s_x$, $c = s_y/s_x$,[3] [4] − [6]. If the estimated standard deviations are taken from (2), then the applicable function is the Elliptic Coverage Function (ELP), $P(R, H, K, s_x, s_y)$. Both functions will be discussed in the next section.[4]

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[2] For clarity, numerical results will be included with the general subject matter.

[3] Note, in general, $s_x \neq s_y$.

[4] The arguments K in $V(K, c)$ and $P(R, H, K, s_x, s_y)$ have different meanings.

II. $V(K, c)$ AND $P(R, H, K, s_x, s_y)$ PROBABILITY FUNCTIONS

The GCE, also known as the elliptical normal function, defines a probability function $V(R/u, v/u)$. GCE gives the probability of a shot falling in a circle of the xy-plane, of radius R and centered at the origin, under a UBND with mean zero and standard deviations u, v. This probability is given by

$$V(R,\ u,\ v) = \frac{1}{2\pi uv} \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \exp\{-\frac{1}{2}[(\frac{x}{u})^2 + (\frac{y}{v})^2]\}\, dy\, dx, \tag{4}$$

where $u = \sigma_x, v = \sigma_y$. Transforming to polar coordinates, with $x = r\cos\theta,\ y = r\sin\theta$ yields

$$V(R,\ u,\ v) = \frac{1}{2\pi uv} \int_{0}^{R} \int_{0}^{2\pi} \exp\{-\frac{1}{2}[(\frac{r\cos\theta}{u})^2 + (\frac{r\sin\theta}{v})^2]\}\, r\, d\theta\, dr. \tag{5}$$

Using elementary trigonometric identities

$$V(R,\ u,\ c) = \frac{1}{\pi c} \int_{0}^{R/u} \int_{0}^{\pi} \exp[-\frac{1}{2}r^2(B - A\cos\theta)]\, r\, d\theta\, dr, \tag{6}$$

with

$$A \equiv \frac{1 - c^2}{2\, c^2}, \qquad\qquad B \equiv \frac{1 + c^2}{2\, c^2}, \tag{7}$$

where, without loss of generality, $v \leq u$,

$$c \equiv v/u, \qquad 0 \leq c \leq 1. \tag{8}$$

Using the fact that

$$I_0(x) \equiv \frac{1}{\pi} \int_{0}^{\pi} \exp(x\cos\theta)\, d\theta, \tag{9}$$

where $I_0(x)$ denotes the modified Bessel function of the first kind and zero order [7, p. 375], one obtains from (6) and (9)

$$V(R/u,\ c) = \frac{1}{c} \int_{0}^{R/u} \exp(-\frac{B\, r^2}{2})\, I_0(\frac{A\, r^2}{2})\, r\, dr. \tag{10}$$

Now letting

$$K \equiv R/u, \qquad w = r^2/2, \tag{11}$$

(10) reduces to

$$V(K,\ c) = \frac{1}{c} \int_{0}^{K^2/2} \exp(-B\, w)\, I_0(A\, w)\, dw. \tag{12}$$

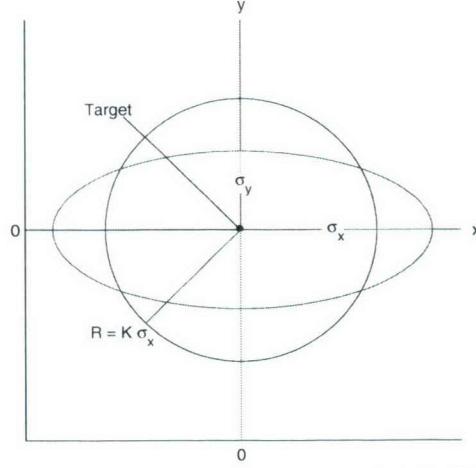The significance of the $V(K, c)$ function in the present context is brought out in Figure 2.

Figure 2. An Elliptical Distribution With Zero Means

Suppose that x and y, the rectangular coordinates of the impact points of a series of weapons, are distributed, each with zero mean, and with standard deviations σ_x and σ_y with $\sigma_y \leq \sigma_x$. The ellipse in Figure 2 has these standard deviations for its semi-axes lengths but has no further significance. Now consider a circle of radius $R = K\sigma_x$ meters, K being a dimensionless ratio. What fraction of the weapons of this distribution hits in the interior of this circle? Or equivalently, as noted earlier, what is the probability that a single weapon from the distribution hits inside the circle? The required probability is given by the function $V(K, c)$ of (10) or (12), where K can take all values from zero to infinity and $0 \leq c \leq 1$. If $c > 1$, then σ_x and σ_y are interchanged with no lose of generality.

The problem considered here is of an inverse character. Estimates s_x and s_y, for σ_x and σ_y, are obtained from bomb drops as noted in (1). Then K is obtained from a subroutine INVGCE, [8], and hence $R(= K s_x)$, such that $V = 0.50$ and $c = s_y / s_x$. The resulting value of R, the radius of the circle, which contains 50% of the distribution, is the computed CEP. Quick estimates for K can be found from the tables included in [4], [5], and [8].

The analysis in [1] is extended here to include nonzero means ($\bar{x} \neq 0$, $\bar{y} \neq 0$). For this purpose, ELP is needed. ELP specifies a probability function $P(R, H, K, u, v)$, [6] and [9], where $u = \sigma_x, v = \sigma_y$, that gives the probability of a shot falling, under a UBND with mean zero and standard deviations u, v, in a circle, T, with radius R and centered at (H, K) of the xy-plane. This probability is given by

$$P(R, H, K, u, v) = \frac{1}{2\pi u\, v} \int_{H-R}^{H+R} \int_{K-\sqrt{R^2-(x-H)^2}}^{K+\sqrt{R^2-(x-H)^2}} \exp\left\{-\frac{1}{2}\left[\left(\frac{x}{u}\right)^2 + \left(\frac{y}{v}\right)^2\right]\right\}\, dy\, dx. \quad (13)$$

4

Introducing dimensionless variables, let $x = (\sqrt{2}\,u)\,s$ and $y = (\sqrt{2}\,v)\,t$, then (13) becomes

$$P(r,\,h,\,k,\,u/v) = \frac{1}{\pi} \int_{h-r}^{h+r} \exp(-s^2) \int_{k-(u/v)\sqrt{r^2-(s-h)^2}}^{k+(u/v)\sqrt{r^2-(s-h)^2}} \exp(-t^2)\,dt\,ds \qquad (14)$$

with $r = R/(\sqrt{2}\,u)$, $h = H/(\sqrt{2}\,u)$, and $k = K/(\sqrt{2}\,v)$. Finally, (14) can be written as

$$P(r,\,h,\,k,\,u/v) = \frac{1}{2\sqrt{\pi}} \int_{h-r}^{h+r} F1(s)\,ds, \qquad (15)$$

where

$$F1(s) \equiv \exp(-s^2)\,\text{aerf}\,[k,\,(u/v)\sqrt{r^2-(s-h)^2}\,], \qquad (16)$$

and

$$\text{aerf}(a,\,b) \equiv \frac{2}{\sqrt{\pi}} \int_{a-b}^{a+b} \exp(-z^2)\,dz. \qquad (17)$$

Equation (15) is used for computational purposes as described in [9], but using simple linear transformations in (13) it is easy to show that ELP can also be interpreted as a probability function $P(R,\,H,\,K,\,u,\,v)$, which gives the probability of a shot falling under a UBND with mean (H, K) and standard deviations u, v, in a circle of radius R centered at the origin of the xy-plane. It is in this context when reference is made to ELP.

The problem considered here, as above for the GCE case, is of an inverse character. Estimates s_x and s_y, for σ_x and σ_y, are obtained from bomb drops as noted in (2). Then R is obtained from the subroutine INVELP, [10], with $P = 0.50$, $u = s_x$, and $v = s_y$. The resulting value of R, the radius of the circle in meters, which contains 50% of the distribution, is the computed CEP for the case of nonzero means. Quick estimates for R can be found from the tables included in [9] and [10].[5]

III. COMPUTATION OF THE ESTIMATED CEP

The data determined from actual bomb drops and used as inputs for computing an estimate of the CEPT are the quantities s_x and s_y, the estimates of the true, but unknown, standard deviations σ_x and σ_y. If n is the number of bombs dropped, and if $(x_1, y_1), \cdots, (x_n, y_n)$ are the coordinates of the n observed impact points (see Figure 1), it is assumed that the x_i's and the y_i's constitute samples from independent normal distributions with means zero or nonzero and standard deviations σ_x and σ_y.

No attempt is made to estimate any correlation coefficient that may be indicated in the data, although such an inclusion would present no difficulty. Also, although s_x^2 and s_y^2 give unbiased estimates for σ_x^2 and σ_y^2, the estimates for σ_x and σ_y given by (1) and (2) are not unbiased; nevertheless, they are invariably used.

Using (1) and (2), we obtain from the data given below (see Figure 1) values for the CEP by using the subroutines INVGCE and INVELP.

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[5]The roles of r and R are interchanged in [10] from their use herein.

The made-up data was obtained from a subroutine that generated normally distributed random numbers with zero mean and $\sigma_x = 35$ and then run again with $\sigma_y = 15$. The data generated is shown in Table 1 and graphically in Figure 1.

Table 1. Normally Distributed Data oF Figure 1, n = 22

i	x_i	y_i	i	x_i	y_i
1	34.281	6.5922	12	-5.7568	23.368
2	-25.493	-5.5825	13	22.839	-16.041
3	-15.329	-31.638	14	-3.3694	-5.9377
4	14.245	4.8120	15	34.301	-5.1021
5	-45.153	.9773	16	3.1264	-7.5556
6	-9.2964	7.4277	17	28.403	.8495
7	47.826	4.5984	18	-14.379	3.2486
8	22.682	-2.8901	19	-48.324	.7353
9	33.730	-16.422	20	-41.711	-19.780
10	-19.193	10.269	21	8.6679	23.298
11	-30.543	-11.832	22	14.292	-18.365

Table 2 contains the quantities required to obtain the listed CEPs using the data from Table 1. The first two columns use (1) with subroutine INVGCE; the last two columns use (2) with subroutine INVELP, where

$$c = s_y/s_x, \quad V = P = 0.50, \quad H = \bar{x}, \quad K = \bar{y}. \tag{18}$$

Table 2. CEPs Based on Table 1 Data

$\bar{x} = 0$	$\bar{y} = 0$	$\bar{x} = 0.26571$	$\bar{y} = -2.4986$
$s_x = 27.592$	$s_y = 13.322$	$s_x = 28.240$	$s_y = 13.393$
$c = .48281$	CEP=23.717	$c = 0.47427$	CEP=24.269

$V = P = PCEP = 0.50$ throughout; but, as mentioned earlier, the subroutine CONFREG of Section VI allows any value in $(0, 1)$ for V or P.

The next section contains an analysis to find CIs for the CEPT.

IV. CONFIDENCE INTERVALS

In this section, we describe three types of confidence regions: conventional (CCI), minimum (MCI), and symmetric (SCI) for the true parameter point (σ_x, σ_y) in half of the quarter plane of the σ_x-σ_y plane $(0 \leq \sigma_y/\sigma_x \leq 1)$.[6] Using these results, the three CIs (CCI, MCI, and SCI) are established for the true value of the CEP (denoted earlier by CEPT or RT, where an estimate is denoted by CEP or R). Hence, our objective is to get meaningful CIs

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[6]It is assumed $\sigma_y \leq \sigma_x$ without loss of generality.

for the CEPT. The approach to achieving this objective is by analyzing the confidence that can be placed in the estimates s_x and s_y, which are quantities deduced directly from the raw experimental data. This Jarnagin idea forms the basis for all results that follow.

Also, the results in actual tests may not appear to be compatible with zero means, but we follow the discussion in [1] and proceed first assuming zero means, which makes the analysis easier to follow. Nevertheless, most of the arguments used also apply to the case of nonzero means except for obvious deviations such as using (2) instead of (1) and using the inverse of ELP, INVELP, instead of the inverse of GCE, INVGCE.

Let a confidence level $\alpha = 1 - \sqrt{PC}$, where confidence regions are determined with an assigned probability, PC; say for illustrative purposes, $PC = (1 - \alpha)^2 \, (= 0.90)$. Then a CI for σ_x with probability $\sqrt{PC} = (1-\alpha) \, (= 0.9486833)$ is obtained by finding appropriate integration limits for the integral of the chi-square density function, the required distribution for the variate $(n\, s_x^2/\sigma_x^2)$ [11, pp. 276-286], where the integral is evaluated at $\alpha/2 \, (= 0.02565835)$ and $(1 - \alpha/2) \, (= 0.97434165)$. Mathematically, we have

$$C(\chi^2_{\alpha/2}, \chi^2_{(1-\alpha/2)}, f) = C(0, \chi^2_{(1-\alpha/2)}, f) - C(0, \chi^2_{\alpha/2}, f), \tag{19}$$

where $C(\gamma, \delta, f)$ denotes the chi-square distribution function with f degrees of freedom, namely

$$C(\gamma, \delta, f) \equiv \frac{1}{2^{f/2}\,\Gamma(f/2)} \int_\gamma^\delta e^{-z/2}\, z^{f/2-1}\, dz, \quad 0 \le \gamma < \delta < \infty. \tag{20}$$

Then $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$, depending on both α and f, are determined by solving the inverse problems

$$C(0, \chi^2_{\alpha/2}, f) = \alpha/2 \, (= 0.02565836), \qquad C(0, \chi^2_{(1-\alpha/2)}, f) = (1 - \alpha/2) \, (= 0.97434165) \tag{21}$$

and by using the subroutine DGINV,[7] [6]. For the numerical examples, using (21), one obtains

$$\chi^2_{\alpha/2} = 11.028377, \quad \chi^2_{(1-\alpha/2)} = 36.677187, \quad \text{using (1) with } f = n = 22, \tag{22}$$

$$\chi^2_{\alpha/2} = 10.327293, \quad \chi^2_{(1-\alpha/2)} = 35.377041, \quad \text{using (2) with } f = n - 1 = 21. \tag{23}$$

Introducing the notation

$$CLx = s_x\sqrt{f/\chi^2_{(1-\alpha/2)}}, \quad CHx = s_x\sqrt{f/\chi^2_{\alpha/2}}, \quad CLy = s_y\sqrt{f/\chi^2_{(1-\alpha/2)}}, \quad CHy = s_y\sqrt{f/\chi^2_{\alpha/2}}, \tag{24}$$

with (22), (23), the CCIs for σ_x are established, [11, p. 276],

$$21.3698 = CLx \le \sigma_x \le CHx = 38.9711, \quad \text{use (1) with } f = n = 22, \tag{25}$$

$$21.7579 = CLx \le \sigma_x \le CHx = 40.2703, \quad \text{use (2) with } f = n - 1 = 21. \tag{26}$$

---

[7]DGINV finds x of the incomplete gamma function, P(a,x), [7, p.260], [12], [13], given P and a, where
$C(0, \chi^2, f) = P(a, x) \equiv \frac{1}{\Gamma(a)} \int_0^x e^{-t}\, t^{a-1}\, dt, \; x = \chi^2/2, \; a = f/2.$

Similarly, CIs are obtained for σ_y using relations in (24), so that,

$$10.3176 = \text{CLy} \leq \sigma_y \leq \text{CHy} = 18.8157, \qquad \text{use (1) with f} = \text{n} = 22, \qquad (27)$$

$$10.3190 = \text{CLy} \leq \sigma_y \leq \text{CHy} = 19.0988, \qquad \text{use (2) with f} = \text{n} - 1 = 21. \qquad (28)$$

We note that a PC of 0.9 may be fairly high; a lower level would result in smaller CIs for σ_x and σ_y.

It is assumed throughout that the α's for the σ_x and σ_y CIs are equal. For the more general situation, they would not be required to be the same. In that case, subroutine CONFREG and a few supporting routines would require a number of changes, but the changes would be straightforward.

In the earlier discussion, the CCI with an assigned probability of $\sqrt{\text{PC}}$ was defined by

$$C(\chi^2_{(1-\alpha/2)}, \infty, f) = C(0, \chi^2_{\alpha/2}, f) = (1 - \sqrt{\text{PC}})/2, \quad C(0, \chi^2_{(1-\alpha/2)}, f) = (1 + \sqrt{\text{PC}})/2. \quad (29)$$

The CIs in this case for σ_x and σ_y are given by (25) and (27) or (26) and (28).

Referring to Figure 3, the vertical stripped region, where (25) holds and σ_y extends from zero to infinity, indicates the $(1-\alpha)$ confidence region for σ_x. The horizontal stripped region, where (27) holds and σ_x extends from zero to infinity, indicates the $(1-\alpha)$ confidence region of σ_y. The intersection of the two regions generates a rectangle, shown in the figure as the crosshatched area, with the lower left-hand corner having the coordinates (CLx, CLy) and the upper right-hand corner with coordinates (CHx, CHy).

Figure 3. 90% Confidence Region For (σ_x, σ_y)

This intersection specifies the PC $(= 90\%)$ conventional confidence region for the point (σ_x, σ_y) based, in general, on the data of which Table 1 is an example. This follows because it is assumed that the x and y coordinates of the bomb drops each arise from *independent* normal distributions with unknown standard deviations, σ_x, σ_y, respectively. Hence, the product of the specified probabilities $(1-\alpha)*(1-\alpha)$ $(= 0.9486833 * 0.9486833 = 0.90) = \text{PC}$.

The rectangular region for the zero means case is shown in Figure 4.

Figure 4. 90% Confidence Region: uses (1), n = 22

Similar arguments result in a (90%) rectangular confidence region for σ_x and σ_y, using (26) and (28), the nonzero means case.

Noting that $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$ are not unique for a given α, they can be varied to obtain, from the subroutine CONFMS, two other CIs, namely MCI (a minimized CI) and SCI (a symmetrized CI). It is understood that $\sqrt{f/\chi^2_{\alpha/2}}$ and $\sqrt{f/\chi^2_{(1-\alpha/2)}}$ are the same for the σ_x-CI and the corresponding σ_y-CI. For MCI, the quantity
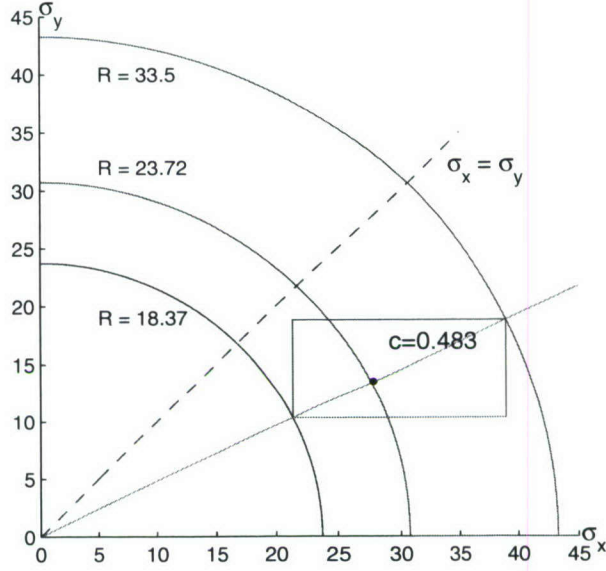
$$\mathrm{CHx} - \mathrm{CLx} \qquad (\text{ see }(24)) \tag{30}$$

is minimized, which means $\mathrm{CHy} - \mathrm{CLy}$ is also minimized. In the case of SCI,[8] the quantity

$$\left|\frac{\mathrm{CHx} + \mathrm{CLx}}{s_x} - 2\right| = \left|\sqrt{f/\chi^2_{\alpha/2}} + \sqrt{f/\chi^2_{(1-\alpha/2)}} - 2\right|, \tag{31}$$

is minimized, which implies

$$\left|\frac{\mathrm{CHy} + \mathrm{CLy}}{s_y} - 2\right| = \left|\sqrt{f/\chi^2_{\alpha/2}} + \sqrt{f/\chi^2_{(1-\alpha/2)}} - 2\right| \tag{32}$$

is also minimized. It should be noted, however, that, for any specific data set, symmetrization is not possible for some $\overline{\alpha}$, if $\sqrt{f/\chi^2_{\overline{\alpha}/2}} \geq 2$, where $C(0, \chi^2_{\overline{\alpha}/2}, f) = \overline{\alpha}/2$.[9]

The results for σ_x and σ_y CIs, using the data from Tables 1 and 2, as well as results from the discussion above, are summarized below in Table 3.

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[8]Note that with (25), the CCI is far from symmetrical; CLx is 23% below s_x, whereas CHx is 41% above s_x.

[9]Since a quantity $b \equiv \sqrt{f/\chi^2_{\alpha}} \leq \sqrt{f/\chi^2_{\overline{\alpha}/2}}$ is needed in the next section, CONFREG requires $b \leq 2$ for symmetrization.

Introducing notation similar to (24), but referring to MCI and SCI with corresponding appropriate values for $\chi^2_{\alpha/2}$ and $\chi^2_{(1-\alpha/2)}$:

$$\text{MLx, SLx} = s_x\sqrt{f/\chi^2_{(1-\alpha/2)}}, \quad \text{MHx, SHx} = s_x\sqrt{f/\chi^2_{\alpha/2}},$$

$$\text{MLy, SLy} = s_y\sqrt{f/\chi^2_{(1-\alpha/2)}}, \quad \text{MHy, SHy} = s_y\sqrt{f/\chi^2_{\alpha/2}}.$$

(33)

Table 3. Listing of CCI, MCI, and SCI Using Data From Table 1

	$\bar{x}=0$	$\bar{x}=0.2657$		$\bar{y}=0$	$\bar{y}=-2.4986$
	s_x=27.5922	s_x=28.2402		s_y=13.3219	s_y=13.3934
CLx	21.3698*	21.7579*	CLy	10.3176*	10.3190*
CHx	38.9711*	40.2703*	CHy	18.8157*	19.0988*
MLx	20.5596	20.8941	MLy	9.9264	9.9094
MHx	37.6151	38.8057	MHy	18.1610	18.4042
SLx	18.3814	18.5228	SLy	8.8748	8.7847
SHx	36.8031	37.9576	SHy	17.7689	18.0020

Table 3 from the 3rd through the 8th rows and the corresponding six columns can be looked at as a 6x6 matrix with elements a(i,j), where, for example, a(1,1)→CLx. Hence, a(1,2) lists the left endpoint of CCI for the $\bar{x}=0$ case; the right endpoint is a(2,2), thus CCI = (21.3698, 38.9711). The MCI for the $\bar{y}=-2.4986$ case is, from the table, (a(3,6), a(4,6)) = (9.9094, 18.4042). Asterisked quantities were given previously in (25) − (28).

We are now in a position, following [1], to determine the CIs for the CEPT. Starting with the zero means case as specified by (25) and (27), consider the $\sigma_x - \sigma_y$ quarter plane ($\sigma_y \geq 0$, $\sigma_x > 0$) shown in Figure 4, and note that nonintersecting curves of constant R, which are concave towards the origin, can be drawn.[10] These constant R-curves, which are not circular segments, are symmetric about the 45 degree line, c = 1, but we need to consider only half of the quarter-plane with $\sigma_y/\sigma_x. \leq 1$.

Then a straight line L emanating from the origin satisfies the equation $\sigma_y = M\sigma_x$, M a constant. For the point $\sigma_x = s_x$, $\sigma_y = s_y$, we have $0 \leq M = c = s_y/s_x \leq 1$. Again, see footnote 10 below; R-curves are definitely *not* segments of circles. Typical curves are shown in Figure 4. They are increasing in R as they move away from the origin. Consider a line L, then since V = 1/2 and c is fixed, it follows K is also constant on L. Therefore, as σ_x increases, moving along L, R must also increase on L to retain K constant on L.

---

[10]These constant R-curves, used only for illustration, can be obtained, referring to the $\sigma_x - \sigma_y$ plane, as follows: For a fixed value of R, determine a value of $K = K_1(= R/\sigma_{x1})$ from $V(K_1,0) = 1/2 = \text{erf}(K_1/\sqrt{2})$, [5, 8, and 7 (p.297)]. This determines the σ_{x1} such that for a given smaller $\sigma_x = \sigma_{x2}$, which fixes a $K = K_2$, a value of c_2 is found by solving the inverse problem $V(K_2,c_2) = 1/2$ that, in turn, determines σ_{y2} thus this is another $(\sigma_{x2}, \sigma_{y2})$ point of the constant R−curve in the $\sigma_x - \sigma_y$ plane.

Note that the point (s_x, s_y) is on the middle R-curve of Figure 4, with R = CEP, intersected by the line with slope c = s_y/s_x = 0.483. In addition, the R-curve containing the point (CLx, CLy), which is also on L, (CLy/CLx = s_y/s_x) has an R-value, CLR, given by solving K = CEP/s_x = 23.717/27.5922 = CLR/CLx. Thus, CLR=18.3685, where CLx is taken from Table 3. Similarly, considering the R-curve passing through (CHx, CHy) on L, with K = CEP/s_x = CHR/CHx, and solving for CHR, one obtains CHR=33.4997. Thus, we take as the CCI for CEPT, CCR ≡ (CLR, CHR) = (18.3685, 33.4997). The MCR and SCR for R are obtained in the same way and defined by MCR ≡ (MLR, MHR) and SCR ≡ (SLR, SHR). The results are summarized in the first and second columns of Table 4.

In the nonzero means case, using (26) and (28), H and K are held fixed throughout. The distribution density function is lower and flatter on each L line as σ_x and σ_y increase with σ_y/σ_x constant; therefore R must increase. Also, R/s_x is not constant along an L line since the normalized variables h, k vary along L, and consequently, R/s_x also varies since P = 0.5; see (14). Therefore CLR(CHR) must be obtained using the subroutine INVELP, with arguments P = 1/2, H = $\bar{x}$, K = $\bar{y}$, σ_x = CLx(CHx), and σ_y = CLy(CHy). Hence the CCR ≡ (CLR, CHR) for the nonzero means case is (18.7754, 34.5007), where, from Table 3, CLx=21.7579, CLy=10.3190, CHx=40.2703, and CHy=19.0988. Note the MCR ≡ (MLR, MHR) for the CEPT in this nonzero case is not precisely minimized, nor is SCR ≡ (SLR, SHR) precisely symmetrized. The overall numerical results are listed in Table 4.

Table 4. Listing of CCI, MCI, and SCI for CEPT
Using Data From Tables 1 and 3

CI↓ Means →	$\bar{x} = 0$, $\bar{y} = 0$	$\bar{x} \neq 0$, $\bar{y} \neq 0$
CLR	18.3685	18.7754
CHR	33.4977	34.5007
MLR	17.6720	18.0454
MHR	32.3322	33.2535
SLR	15.7998	16.0451
SHR	31.6342	32.5316

Recalling the second objective (see page 1, 2nd paragraph) of establishing whether the design contractor of the RBD is to be rewarded or penalized, two other confidence regions are introduced. They form the discussion in the next section with the idea of sequencing bomb drops. We follow the reasoning in [1].

V. CONFIDENCE INTERVALS BY SEQUENTIAL TESTING

Let us assume we are dealing with the zero means case: (25) and (27). Begin by assigning RTn, an agreed nominal value for the CEP, and a fixed-decimal FP. Then assume that if the CEP $\leq (1 - \text{FP})\text{RTn} = \text{RN}$, within the assigned confidence level, the contractor is rewarded (COR) and if CEP $\geq (1+\text{FP})\text{RTn} = \text{RP}$, the contractor is penalized (COP). With a numerical example, suppose FP $= 0.10$, RTn $= 40$; then from Table 2, CEP $= 23.717 < 36$; can we conclude the contractor should be rewarded? The answer is "yes" because, from Table 4, 36 is to the right of the interval (18.3685, 33.4977), equivalently greater than CHR, which in 90% of similar test cases will contain the CEPT and will be less than 36. Instead, suppose the RTn $= 16$. Since the CEP $= 23.717$ is greater, we try to judge if the contractor should be penalized. Namely, is $16 * 1.1 = 17.6 < \text{CLR}(= 18.3685)$? The answer is again "yes" in the numerical example, since in 90% of repeated like tests the CEPT would exceed 17.6. In either case, if the answer was negative due to a different choice of RTn, then more testing would be indicated (MOT).

Before considering the procedure of more testing, it is important to arrive at a conclusion with a minimum of testing. Hence we ask in the COR case if we can lower CHR; i.e., ease the requirement for a decision. Therefore, using DGINV, we ask for the value of χ^2_α such that

$$C(\chi^2_\alpha, \infty, f) = 1 - \alpha = 0.9486833, \quad C(0, \chi^2_\alpha, f) = \alpha = 0.0513167, \tag{34}$$

so that

$$f\, s_x^2/\sigma_x^2 \geq \chi^2_\alpha \Rightarrow \text{ the CI for } \sigma_x: \quad 0 < \sigma_x \leq b\, s_x, \quad \sigma_y \geq 0, \quad b \equiv \sqrt{f/\chi^2_\alpha} \geq 1. \tag{35}$$

$$f\, s_y^2/\sigma_y^2 \geq \chi^2_\alpha \Rightarrow \text{ the CI for } \sigma_y: \quad 0 < \sigma_y \leq b\, s_y, \quad \sigma_x > 0, \quad b \equiv \sqrt{f/\chi^2_\alpha} \geq 1. \tag{36}$$

The intersection of the two infinite CIs is shown as the hatched region in Figure 5.

Note then, as in Section IV, the $(1 - \alpha)^2$ CI for the CEPT is given by solving for $\underline{\text{CHR}}$

$$K = \text{CEP}/s_x = \underline{\text{CHR}}/(b\, s_x) \Rightarrow \underline{\text{CHR}} = b\, \text{CEP}, \tag{37}$$

where, in general,

$$\underline{\text{CHR}} \leq \text{CHR (see Section IV for CHR)}. \tag{38}$$

In the case where (2) holds, INVELP with standard deviations $b\, s_x$ and $b\, s_y$ is used to find $\underline{\text{CHR}}$. For the same reasons given in the previous section in the discussion of the nonzero means case, (37) cannot be used (See the paragraph before Table 4.).

We ask in the COP case if we can raise CLR; i.e., ease the requirement for a decision. Therefore, using DGINV, we ask for the value of $\chi^2_{(1-\alpha)}$ such that

$$C(0, \chi^2_{(1-\alpha)}, f) = 1 - \alpha = 0.9486833, \tag{39}$$

so that

$$f\, s_x^2/\sigma_x^2 \leq \chi^2_{(1-\alpha)} \Rightarrow \text{ the CI for } \sigma_x: \quad 0 < a\, s_x \leq \sigma_x, \quad \sigma_y \geq 0, \quad a \equiv \sqrt{f/\chi^2_{(1-\alpha)}} \leq 1. \tag{40}$$

$$f\, s_y^2/\sigma_y^2 \leq \chi^2_{(1-\alpha)} \Rightarrow \text{ the CI for } \sigma_y: \quad 0 < a\, s_y \leq \sigma_y, \quad \sigma_x > 0, \quad a \equiv \sqrt{f/\chi^2_{(1-\alpha)}} \leq 1. \tag{41}$$

The intersection of the two infinite CIs is shown as the hatched region in Figure 6.

Note then, as in Section IV, the $(1-\alpha)^2$ CI for the CEPT is given by solving for $\overline{\text{CLR}}$

$$K = \text{CEP}/s_x = \overline{\text{CLR}}/(a\,s_x) \;\Rightarrow\; \overline{\text{CLR}} = a\,\text{CEP}, \tag{42}$$

where, in general,

$$\text{CLR} \leq \overline{\text{CLR}} \text{ (see Section IV for CLR)}. \tag{43}$$

In the case where (2) is used instead of (1), INVELP with standard deviations $a\,s_x$ and $a\,s_y$ is used to find $\overline{\text{CLR}}$. Equation (42) cannot be used for the same reasons given in the previous section; (see the paragraph before Table 4.)
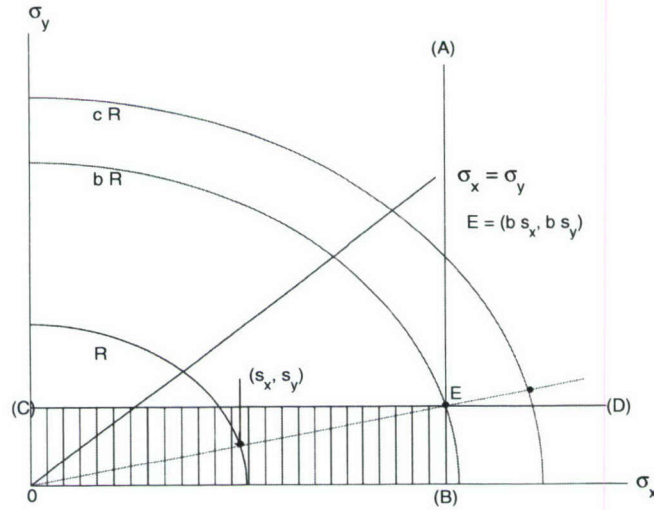

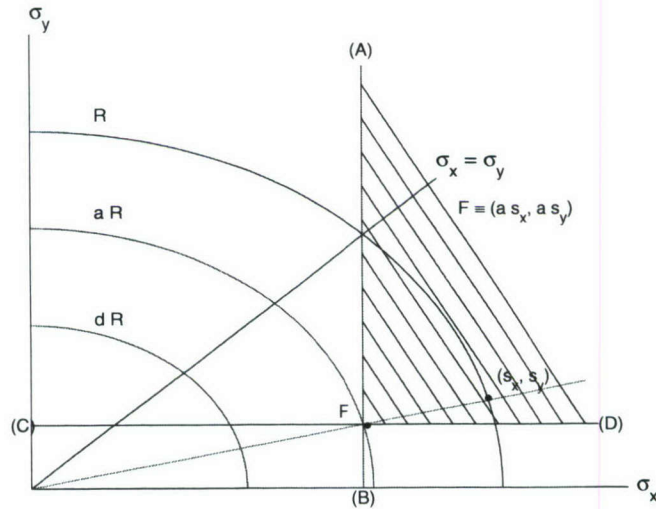Figure 5. With Confidence $(1-\alpha)^2$, $\;0 \leq \text{RT} < b\,R$

Figure 6. With Confidence $(1-\alpha)^2$, $\;a\,R \leq \text{RT} < \infty$

Table 5 Contains$\overline{\text{CLR}}$ And $\underline{\text{CHR}}$; It Supplements Table 4.

Table 5. Listing of $\overline{\text{CLR}} \geq$ CLR, $\quad \underline{\text{CHR}} \leq$ CHR
Using Data From Tables 1 and 4

CI↓ Means →	$\bar{x} = 0$, $\bar{y} = 0$	$\bar{x} \neq 0$, $\bar{y} \neq 0$
$\overline{\text{CLR}}$	19.1306	19.5547
$\underline{\text{CHR}}$	31.5975	32.5010

Perhaps the most efficient strategy in what has been discussed so far, to evaluate whether COR, COP, or MOT holds, is to determine $\overline{\text{CLR}}$ or $\underline{\text{CHR}}$ at each new set of sequential bombing tests. Given RTn, FP, n0, Δn, and Jmax, where n0 denotes the initial number of bomb drops, Δn is the increment number of bomb drops added at each test after the initial n0 drops, and Jmax denotes the assigned number of Δns, incremental bomb drops to be made, after which all testing stops.

Using the data from Tables 1 and 2, with (1), CEP = 23.717 and with (2), CEP = 24.269, assign the following:

$$n0 = 10, \quad \Delta n = 4, \quad Jmax = 3. \tag{44}$$

Tests, as described in Section 4 and this section, are conducted for each n, where

$$n = n0 + j\Delta n, \quad j = 0, 1, \cdots, Jmax. \tag{45}$$

The results are given in Table 6 of page 15. CIs can be identified from the table accordingly:

1. Conventional $\quad (1 - \alpha)^2 (= .90)$ CI for the CEPT: $(L(1), H(1)) = (\text{CLR, CHR})$
2. Minimum $\quad (1 - \alpha)^2 (= .90)$ CI for the CEPT: $(L(2), H(2)) = (\text{MLR, MHR})$
3. Symmetric $\quad (1 - \alpha)^2 (= .90)$ CI for the CEPT: $(L(3), H(3)) = (\text{SLR, SHR})$
4. Maximum CLR $\quad (1 - \alpha)^2 (= .90)$ CI for the CEPT: $(L(4), H(4)) = (\overline{\text{CLR}}, \infty)$
5. Minimum CHR $\quad (1 - \alpha)^2 (= .90)$ CI for the CEPT: $(L(5), H(5)) = (0, \underline{\text{CHR}})$

For example, for the zero means case, for the symmetric CI at n = 18, one obtains from the table (13.7266, 30.6586) with a CEP = 22.1926; the minimum CHR CI is (0, 30.6462). For the nonzero means case, one obtains for the symmetric CI at n = 18, (14.0863, 31.6683) with a CEP = 22.8205; the minimum CHR = CI is (0, 31.6598).

If RTn = 16.5 and FP = 0.10, then COP holds for the zero means case, since $18.15 < \overline{\text{CLR}} = 19.1306 <$ CEP at n = 22. In the nonzero means case, COP also holds since $18.15 < \overline{\text{CLR}} = 18.8265 <$ CEP for n = 10. On the other hand, let RTn = 34.2 and FP = 0.1, then COR holds in the zero means case, since $\underline{\text{CHR}} = 30.6462 < 30.78$ for n = 18, but in the nonzero means cases MOT holds, since for n = 10, 14, 18, 22, $\underline{\text{CHR}} > 30.78$.

In the unlikely event that RP $\approx$ CEPT or RN $\approx$ CEPT, it may require a very large value of n to conclude whether COP or COR holds.

Table 6. Listing of CIs for CEPT–Using Data From Previous Tables

L(1)=CLR, L(2)=MLR, L(3)=SLR, L(4)=$\overline{\text{CLR}}$, L(5)=0							
H(1)=CHR, H(2)=MHR, H(3)=SHR, H(4)=∞, H(5)=$\underline{\text{CHR}}$							
j, n, m	L(m)	H(m)	CEP	j, n, m	L(m)	H(m)	CEP
$\overline{x}$	$\overline{y}$	s_x	s_y	$\overline{x}$	$\overline{y}$	s_x	s_y
0.0	0.0	29.4891	12.4880	3.8299	-2.1856	30.8210	12.9603
0, 10, 1	16.9769	42.4081	24.2501	0, 10, 1	17.7882	46.1287	25.5357
0, 10, 2	15.6089	39.3400	24.2501	0, 10, 2	16.2745	42.4788	25.5357
0, 10, 3	10.0136	38.4866	24.2501	0, 10, 3	9.8215	41.5872	25.5357
0, 10, 4	17.9640	∞	24.2501	0, 10, 4	18.8265	∞	25.5357
0, 10, 5	0.0	38.4866	24.2501	0, 10, 5	0.0	41.5872	25.5357
$\overline{x}$	$\overline{y}$	s_x	sy	$\overline{x}$	$\overline{y}$	s_x	s_y
0.0	0.0	26.9855	13.4646	1.5335	-2.3071	27.9590	13.7662
1, 14, 1	17.2129	36.9126	23.4708	1, 14, 1	17.7751	38.9652	24.3463
1, 14, 2	16.2033	34.9524	23.4708	1, 14, 2	16.6815	36.7593	24.3463
1, 14, 3	12.7839	34.1578	23.4708	1, 14, 3	12.8455	35.9331	24.3463
1, 14, 4	18.0811	∞	23.4708	1, 14, 4	18.6803	∞	24.3463
1, 14, 5	0.0	34.1564	23.4708	1, 14, 5	0.0	35.9326	24.3463
$\overline{x}$	$\overline{y}$	s_x	sy	$\overline{x}$	$\overline{y}$	s_x	s_y
0.0	0.0	26.2413	12.0934	4.0512	-2.2699	26.6783	12.2229
2, 18, 1	16.7947	32.7412	22.1926	2, 18, 1	17.3269	33.8762	22.8205
2, 18, 2	16.0203	31.3648	22.1926	2, 18, 2	16.5234	32.3927	22.8205
2, 18, 3	13.7266	30.6586	22.1926	2, 18, 3	14.0863	31.6683	22.8205
2, 18, 4	17.5552	∞	22.1926	2, 18, 4	18.0945	∞	22.8205
2, 18, 5	0.0	30.6462	22.1926	2, 18, 5	0.0	31.6598	22.8205
$\overline{x}$	$\overline{y}$	s_x	sy	$\overline{x}$	$\overline{y}$	s_x	s_y
0.0	0.0	27.5922	13.3219	0.2657	-2.4986	28.2403	13.3934
3, 22, 1	18.3685	33.4977	23.7170	3, 22, 1	18.7754	34.5007	24.2689
3, 22, 2	17.6720	32.3322	23.7170	3, 22, 2	18.0454	33.2535	24.2689
3, 22, 3	15.7998	31.6342	23.7170	3, 22, 3	16.0451	32.5316	24.2689
3, 22, 4	19.1306	∞	23.7170	3, 22, 4	19.5547	∞	24.2689
3, 22, 5	0.0	31.5975	23.7170	3, 22, 5	0.0	32.5010	24.2689

The next section describes the Fortran 95 computer program that outputs the results of Table 6 as well as whether MOT or COP or COR holds given RN, RP.

VI. FORTRAN PROGRAMS FOR CEP AND CONFIDENCE INTERVALS

The total software package is made up of 69 double-precision Fortran 95 subprograms. Each of their *call lines* is listed in Appendix B. The first 14 were developed by the author; the remaining 55 are supporting routines from the NSWC Library of Mathematics Subroutines, [6]. Of the first 14, we shall discuss the first 5; the remaining 9 are supporting routines. The software is stored in a file named WEEV4.FOR.

The Program routine, MEANSTDV, is actually not necessary and can be deleted. It is a test routine for the software that uses, as input, a file JDATA. If the user wishes, the software could be run with a read-in input of 0.5 or 0.9, and 2 when requested, and directing the results by WEEV4 > f8; then making a DOS file "compare," FC f8 JDATATEST. The files should be duplicates except for the last line, which calls on the Lahey subroutines DATE and TIME. The required JDATA on the first line would list the two integers 1 and 22, on the 2nd line store x_1, y_1, and the following 21 lines would contain the remaining miss distances of Table 1, x_i, y_i, $(i = 2, 22)$.

The 2nd listed routine,

CONFREG(N,X,Y,PCEP,PC,ISV,RN,RP,XYM,SXY,RCEP,RLH,DEC),

is the main routine for WEEV4, and it is the only one a user needs to call. Its call line specifies the necessary input and output. The first ten variables are required input; the last three are output.

INPUT

N	Number of bomb drops.
X	N-element array containing the x-coordinate miss distances.
Y	N-element array containing the y-coordinate miss distances.
PCEP	Probability measure for the CEPT. Usually taken as 0.50.
PC	Assigned confidence level for CIs. Taken as 0.9 in the examples.
ISV	If $0(> 0)$, then do zero(nonzero) means case.
RN	Value $(1 - FP)\,RTn$. In computer numerical example $FP = 0.125$, $RTn = 16\,2/3$, $RN = 14.583333$. If $RN < 0$, the contractor evaluation is skipped.
RP	Value $(1 + FP)\,RTn$. In computer example $RP = 18.75$.
XYM	Two-element array containing means $\bar{x}$ and $\bar{y}$.
SXY	Two-element array containing standard deviations s_x and s_y.

OUTPUT

RCEP   Estimated CEP. If ISV = 0 uses INVGCE, else uses INVELP.

RLH   Eight-element array with the following elements (See Table 6):

       RLH(1) − CLR, Left endpoint of the conventional CI for CEPT.

       RLH(2) − CHR, Right endpoint of the conventional CI for CEPT.

       RLH(3) − MLR, Left endpoint of the minimum CI for CEPT.

       RLH(4) − MHR, Right endpoint of the minimum CI for CEPT.

       RLH(5) − SLR, Left endpoint of the symmetric CI for CEPT.

       RLH(6) − SHR, Right endpoint of the symmetric CI for CEPT.

         If $\underline{CHR} > 2$, then SLR and SHR set to −1.

       RLH(7) − $\overline{CLR}$, Maximum CLR, left endpoint of a CI for CEPT.

       RLH(8) − $\underline{CHR}$, Minimum CHR, right endpoint of a CI for CEPT.

DEC   A 3-character variable specifying contractor performance:

       COR, COP, MOT; if input RN < 0, ignore DEC.

The next three routines are called by CONFREG. MNSTD is used to compute $\bar{x}$, $\bar{y}$, and s_x, s_y. CONFMS is used to find the three CIs: namely, CCR, MCR, and SCR. EVAL is used to determine if the contractor is to be rewarded, COR; if the contractor is to be penalized, COP; or if more testing is needed, MOT.

VII. REFERENCES

1. Jarnagin, M. P., *Computation of the CEP and CIs in Bombing Tests*, Naval Weapons Laboratory, Dahlgren, VA, May 1971. Informal Memorandum

2. Taub, A. E. and Thomas, M. A., *CIs for CEP Where Errors are Elliptical Normal*, NSWC TR 83-205, November 1983, Naval Surface Weapons Center, Dahlgren, VA.

3. Thomas, M. A. and Taub, A. E., *Weapon Accuracy Assessment for Elliptical Normal Miss Distances*, NSWC/DL TR-3777, January 1978, Naval Surface Weapons Center, Dahlgren, VA.

4. Weingarten, H. and DiDonato, A. R., "A Table of Generalized Circular Error," *Mathematics of Computation,* Vol. 15, No. 74, April 1961, 169-173.

5. DiDonato, A. R. and Jarnagin, M. P., *A Method for Computing the Generalized Circular Error Function and the Circular Coverage Function*, NWL Report No. 1768, January 1962, Naval Weapons Laboratory, Dahlgren, VA.

6. Morris, A. H., *NSWC Library of Mathematics Subroutines*, NSWCDD/TR-92/425, January 1993, Naval Surface Warfare Center, Dahlgren Division, Dahlgren, VA.

7. Abramowitz, M. and Stegun, I., (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* Applied Mathematics Series 55, June 1955.

8. DiDonato, A. R., *An Inverse of the Generalized Circular Error Function*, NSWCDD/TR-04/43, June 2004, Naval Surface Warfare Center, Dahlgren, VA.

9. DiDonato, A. R. and Jarnagin, M. P., *Integration of the General Bivariate Gaussian Distribution over an Offset Ellipse*, NWL Report No. 1710, August 1960, Naval Weapons Laboratory, Dahlgren, VA.

10. DiDonato, A. R., *An Inverse of the Elliptic Coverage Function*, NSWCDD/TR-05/90, April 2005, Naval Surface Warfare Center, Dahlgren, VA.

11. Hald, A., *Statistical Theory with Engineering Applications*, John Wiley & Sons, Inc.,1952.

12. DiDonato, A. R. and Morris, A. H., "Computation of the Incomplete Gamma Function Ratios and Their Inverse," *ACM Trans. Math Software,* Vol. 12, 1986, 377-393.

13. DiDonato, A. R. and Hageman, R. K., *Computation of the Percentage Points of the Chi-Square Distribution*, NSWCDDL TR-3569, 1977, Naval Surface Weapons Center, Dahlgren, VA.

APPENDIX A

GLOSSARY

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APPENDIX A

GLOSSARY

The first number listed with each item, refers to the page where it is first introduced.

$C(\gamma, \delta, f)$	-	Integral of the Chi-square distribution density from γ to δ, 7
$\underline{CHR}$	-	The minimum value CHR can take for a given α, 12
$\overline{CLR}$	-	The maximum value CLR can take for a given α, 13
CEPT	-	True CEP, 1, 6
α	-	Confidence level, 7, 9
Δn	-	Increment number of bomb drops added at each test after the first, 14
$(1 - \alpha)$	-	$(1 - \alpha) = \sqrt{PC}$, 7
CCI	-	Conventional confidence interval, 6
CCR	-	Conventional confidence interval for CEPT, 11
CEP	-	Estimated circular error probable, 1
CHR	-	R, Right endpoint of the CCI, CCR, for CEPT, 11
CHx	-	Right endpoint of the CCI for σ_x. See (24), 7
CHy	-	Right endpoint of the CCI for σ_y. See (24), 7
CI	-	Confidence interval(s), 1, 8
CLR	-	R, Left endpoint of the CCI, CCR, for CEPT, 11
CLx	-	Left endpoint of the CCI for σ_x. See (24), 7
CLy	-	Left endpoint of the CCI for σ_y. See (24), 7
COP	-	Contractor is penalized, 12
COR	-	Contractor is rewarded, 12
ELP	-	Elliptic Coverage Function, 2
ELP	-	ELP subroutine to compute ELP, 5
f	-	number of degrees of freedom, 2
FP	-	A fixed decimal associated with RTn, 11
GCE	-	GCE subroutine to evaluate GCE, 3
GCE	-	Generalized Circular Error Function, 2
INVELP	-	Subroutine to compute R of $P(R, H, K, s_x, s_y)$, 5
INVGCE	-	Subroutine to compute K of $V(K,c)$, 4
Jmax	-	Maximum number of Δns, 14
K	-	Dimensionless radius of GCE. $K = R/u$, 3
K	-	y-coordinate of the mean of the Elliptic Coverage Function, 5

MCI	-	Minimized confidence interval, 6, 9
MCR	-	Minimum confidence interval for CEPT, 11
MHR	-	R, Right endpoint of the MCI, MCR, for CEPT, 11
MHx	-	Right endpoint of the MCI for σ_x. See (33), 10
MHy	-	Right endpoint of the MCI for σ_y. See (33), 10
MLR	-	R, Left endpoint of the MCI, MCR, for CEPT, 11
MLx	-	Left endpoint of the MCI for σ_x. See (33), 10
MLy	-	Left endpoint of the MCI for σ_y. See (33), 10
MOT	-	More testing required, 12
n0	-	Initial number of bomb drops, 14
PC	-	Assigned probability for the confidence interval of the CEPT, 7
PCEP	-	Percentage/100 of the UBND contained in the target circle, 1
$P(R, H, K, s_x, s_y)$	-	Elliptic Coverage Function, 2
R	-	Estimated CEP (computed), 1
R	-	Used interchangeably with the estimated CEP, 2
RBD	-	Radar bomb director, 1
RN	-	=(1-FP)RTn, 12
RP	-	=(1+FP)RTn, 12
RT	-	True CEP=CEPT, 1
RTn	-	A pre specified nominal value for the CEP, 1
SCI	-	Symmetrical confidence interval, 6, 9
SCR	-	Symmetrical confidence interval for CEPT, 11
SHR	-	R, Right endpoint of the SCI, SCR, for CEPT, 11
SHx	-	Right endpoint of the SCI for σ_x. See (33), 10
SHy	-	Right endpoint of the SCI for σ_y. See (33), 10
SLR	-	R, Left endpoint of the SCI, SCR, for CEPT, 11
SLx	-	Left endpoint of the SCI for σ_x. See (33), 10
SLy	-	Left endpoint of the SCI for σ_y. See (33), 10
UBND	-	Uncorrelated Bivariate Normal Distribution, 1
V(K, c)	-	Generalized Circular Error Function, 2

APPENDIX B

LIST OF FORTRAN 95 SUBPROGRAMS

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APPENDIX B

LIST OF FORTRAN 95 SUBPROGRAMS

1 PROGRAM MEANSTDV

2 SUBROUTINE CONFREG(N,X,Y,PCEP,PC,ISV,RN,RP,XYM,SXY,RCEP,RLH,DEC)

3 SUBROUTINE MNSTD(N,X,XM,VX2,SX2)

4 SUBROUTINE CONFMS(N,P,RL0,RH0,RL1,RH1,RL2,RH2)

5 SUBROUTINE EVAL(N,RN,RP,CHR)

6 DOUBLE PRECISION FUNCTION FD(X) !FOR MINIMUM CONF INTERVALS,
USED IN FMIN IN CONFMS

7 DOUBLE PRECISION FUNCTION FA(X) !FOR SYMMETRIC CONF INTERVALS,
USED IN FMIN IN CONFMS

8 SUBROUTINE INVGCE(P,C,R,IJ,IERR)

9 SUBROUTINE INVELP(PP,H1,K1,S1,S2,R,P,I)

10 DOUBLE PRECISION FUNCTION F2(X) !NEEDED FOR DZERO OF INVELP

11 SUBROUTINE DQ(R,PX,II)

12 DOUBLE PRECISION FUNCTION F1(X) !FUNCTION FOR DQXGS ROUTINE

13 SUBROUTINE GRUB(R) !INITIAL ESTIMATE FOR R FROM GRUBBS' APPROX

14 SUBROUTINE PSQR(A,P,X,I) !1ST APPROX, X, USES GRUBBS' ESTIMATE

15 INTEGER FUNCTION IPMPAR (I)

16 DOUBLE PRECISION FUNCTION DPMPAR (I)

17 DOUBLE PRECISION FUNCTION DEPSLN (L)

18 DOUBLE PRECISION FUNCTION DXPARG (L)

19 DOUBLE PRECISION FUNCTION DSIN1 (X)

20 DOUBLE PRECISION FUNCTION REXP (X)

21 DOUBLE PRECISION FUNCTION DREXP (X)

22 DOUBLE PRECISION FUNCTION ALNREL(A)

23 DOUBLE PRECISION FUNCTION DLNREL (A)

24 DOUBLE PRECISION FUNCTION RLOG(X)

25 DOUBLE PRECISION FUNCTION DRLOG (X)

26 DOUBLE PRECISION FUNCTION ERF (X)

27 DOUBLE PRECISION FUNCTION ERFC1 (IND, X)

28 DOUBLE PRECISION FUNCTION DERF (X)

29 DOUBLE PRECISION FUNCTION DERFC (X)

30 DOUBLE PRECISION FUNCTION DERFC1 (IND, X)

31 DOUBLE PRECISION FUNCTION DERFC0 (X)

32 DOUBLE PRECISION FUNCTION ERFI (P, Q)

33 DOUBLE PRECISION FUNCTION DERFI (P, Q)

34 DOUBLE PRECISION FUNCTION DAERF (X, H)

35 SUBROUTINE PNI (P, Q, D, W, IERR)

36 SUBROUTINE DPNI (P, Q, D, W, IERR)

37 DOUBLE PRECISION FUNCTION GAMMA(A)

38 DOUBLE PRECISION FUNCTION GLOG(X)

39 DOUBLE PRECISION FUNCTION GAM1(A)

40 DOUBLE PRECISION FUNCTION GAMLN (A)

41 DOUBLE PRECISION FUNCTION GAMLN1 (A)

42 DOUBLE PRECISION FUNCTION DGAMMA(A)

43 DOUBLE PRECISION FUNCTION DPDEL(X)

44 DOUBLE PRECISION FUNCTION DGAM1 (X)

45 DOUBLE PRECISION FUNCTION DGAMLN (A)

46 DOUBLE PRECISION FUNCTION DGMLN1 (X)

47 SUBROUTINE GRATIO (A, X, ANS, QANS, IND)

48 DOUBLE PRECISION FUNCTION RCOMP (A, X)

49 DOUBLE PRECISION FUNCTION DRCOMP (A, X)

50 SUBROUTINE GAMINV (A, X, X0, P, Q, IERR)

51 SUBROUTINE DGRAT (A, X, ANS, QANS, IERR)

52 SUBROUTINE DGR29 (A, Y, L, Z, RTA, ANS, QANS)

53 SUBROUTINE DGR17 (A, Y, L, Z, RTA, ANS, QANS)

54 SUBROUTINE DGINV (A, X, P, Q, IERR)

55 SUBROUTINE BESI(X, ALPHA, KODE, N, Y, NZ)

56 SUBROUTINE ASIK(X,FNU,KODE,FLGIK,RA,ARG,IN,TOL,Y)

57 SUBROUTINE CIRCV (R, D, J, P, IERR)

58 SUBROUTINE ERFC0 (IND, X, E, Y)

59 DOUBLE PRECISION FUNCTION ERFCR (X)

60 DOUBLE PRECISION FUNCTION DZERO (F, AX, BX, AERR, RERR)

61 SUBROUTINE FMIN(F, A0, B0, X, W, AERR, RERR, ERROR, IND)

62 SUBROUTINE DQPSRT(LIMIT,LAST,MAXERR,ERMAX,ELIST,IORD,NRMAX)

63 SUBROUTINE DQELG (N, EPSTAB, RESULT, ABSERR, RES3LA, NRES,
            EPMACH, OFLOW)

64 SUBROUTINE DQXGS (F,A,B,EPSABS,EPSREL,RESULT,ABSERR,IER,
            LIMIT,LENIW,LENW,LAST,IWORK,WORK)

65 SUBROUTINE DQXGSE(F,A,B,EPSABS,EPSREL,LIMIT,RESULT,ABSERR,
            IER,ALIST,BLIST,RLIST,ELIST,IORD,LAST,VALP,VALN,LP,LN)

66 SUBROUTINE DQXCPY (A, B, L)

67 SUBROUTINE DQXLQM (F,A,B,RESULT,ABSERR,RESABS,RESASC,VR,VS,LR,LS,
KEY,EPMACH,UFLOW,OFLOW)

68 SUBROUTINE DQXRUL (F,XL,XU,Y,YA,YM,KE,K1,FV1,FV2,L1,L2)

69 SUBROUTINE DQXRRD (F,Z,LZ,XL,XU,R,S,LR,LS)

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